PRACTICE FINAL (FROEHLE) - SOLUTIONS

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1. - Domain: $\mathbb{R}$
   - Intercepts: 0 is both an $x-$ intercept and a $y-$ intercept
   - Symmetries: None
   - Asymptotes: $y = 0$ is a H.A. at $-\infty$ because $\lim_{x\to-\infty} f(x) = 0$ (by l'Hopital's rule). Those are all of the asymptotes!
   - Increasing/Decreasing: $f'(x) = (x+1)e^x$, f is decreasing on $(-\infty, -1)$ and increasing on $(-1, \infty)$, Local minimum $(-1, -e^{-1})$
   - Concavity: $f''(x) = (x+2)e^x$, f is concave down on $(-\infty, -2)$, concave up on $(-2, \infty)$, Inflection point $(-2, -2e^{-2})$
   - Graph: Whip out your calculator to see what the graph looks like!

2. Let's first simplify B using the substitution $u = x + 1$ (then $du = dx$, and $x = u - 1$:

$$B = \int_0^3 2x\ln(x+1)dx = \int_1^4 2(u-1)\ln(u)du = \int_1^4 2(x-1)\ln(x)dx = C$$

So $\boxed{B = C}$

Now, let's use the substitution $u = \sqrt{x}$ to simplify D, (then $du = \frac{1}{2\sqrt{x}}dx$, so $dx = 2\sqrt{x}du = 2udu$), so:

$$D = \int_0^9 \ln(\sqrt{x}+1)dx = \int_0^3 2u\ln(u+1)du = \int_0^3 2x\ln(x+1)dx = B$$

So $\boxed{D = B}$.

Hence, our final answer is $\boxed{B = C = D}$ (and those are all different from A because for example $A \neq C$)

3. Don't worry about this one, it has to do with the mean value theorem for integrals (section 6.5) , for which you are not responsible!

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4. (a) You can do that using either the disk method or the shell method!

Shell method: $x = 2$, so $K = 2$, so Radius $=|x-2| = 2-x$ (since $x < 2$, Peyam method FTW!!!) , and Height = x^3, hence:

$$V = \int_1^2 2\pi(2-x)x^3 dx = 2\pi \int_1^2 2x^3 - x^4 dx = 2\pi \left[\frac{x^4}{2} - \frac{x^5}{5}\right]_1^2 = 2\pi(8 - \frac{32}{5} - \frac{1}{2} + \frac{1}{5}) = \frac{13\pi}{5}$$

Disk method: also works, but you'd have to solve for x in terms of y, your radius becomes Rightmost - Leftmost = $2 - y^{\frac{1}{3}}$, and your volume becomes $V = \int_1^8 \pi \left(2 - y^{\frac{1}{3}}\right)^2 dy = \frac{8\pi}{5}$

(b) This is a typical application of the disk method:

$$V = \int_1^2 \pi \left(\frac{1}{x}\right)^2 dx = \pi \left[-\frac{1}{x}\right]_1^2 = \pi(1 - \frac{1}{2}) = \frac{\pi}{2}$$

5. (a) Let $u = 1 + x^4$, then $du = 4x^3 dx$, so $x^3 dx = \frac{1}{4} du$, hence:

$$\int \frac{x^3}{1+x^4} dx = \int \frac{1}{u}\frac{1}{4} du = \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln(1+x^4) + C$$

The absolute values don't matter here because $1 + x^4 > 0$

(b) Let $u = \ln(\cos(x))$, then $du = -\frac{\sin(x)}{\cos(x)} dx = -\tan(x) dx$, so:

$$\int \tan(x) \ln(\cos(x))\, dx = \int -u du = -\frac{u^2}{2} + C = -\frac{(\ln(\cos(x)))^2}{2} + C$$

Note: You could also have used the substitution $u = \cos(x)$, but then you'd have to use another substitution $v = \ln(u)$ to evaluate the resulting integral!

(c) The trick here is to expand everything out, namely $(1-x)^2 = 1 - 2x + x^2$, so:

$$\int \left(\frac{1-x}{x}\right)^2 dx = \int \frac{1-2x+x^2}{x^2} dx = \int \frac{1}{x^2} - \frac{2}{x} + 1 dx = -\frac{1}{x} - 2\ln|x| + x + C$$

6. Let G be an antiderivative of $g(t) = \frac{e^t}{t}$. Then $f(x) = G(x) - G(\sqrt{x})$, so:

$$f'(x) = G'(x) - G'(\sqrt{x})\frac{1}{2\sqrt{x}} = g(x) - g(\sqrt{x})\frac{1}{2\sqrt{x}} = \frac{e^x}{x} - \frac{e^{\sqrt{x}}}{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right) = \frac{e^x}{x} - \frac{1}{2}\frac{e^{\sqrt{x}}}{x}$$

7. Do **NOT** try to evaluate that integral, it'll drive you crazy! Plus, there's a theorem that says that you cannot explicitly write an antiderivative of e^{-x^2}.

   But notice the following: Since $0 \leq x \leq 1$, $-1 \leq -x^2 \leq 0$, so $e^{-1} \leq e^{-x^2} \leq e^0 = 1$ (e^x is increasing), so, integrating, we get: $\int_0^1 e^{-1} dx \leq \int_0^1 e^{-x^2} dx \leq \int_0^1 1 dx$. But $\int_0^1 e^{-1} dx = e^{-1} \int_0^1 1 dx = e^{-1}$ and $\int_0^1 1 dx = 1$, so we get:

$$e^{-1} \leq int_0^1 e^{-x^2} dx \leq 1$$

   Which is what we wanted to show!

8. $\Delta x = \frac{1}{n}$, $x_i = a + i\Delta x = \frac{i}{n}$

$$\begin{aligned}
\int_0^1 x^2 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3} \\
&= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} \\
&= \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \\
&= \frac{2}{6} \\
&= \frac{1}{3}
\end{aligned}$$

9. The tangent line to $y = cx^2$ has slope $\boxed{A = \frac{dy}{dx} = 2cx}$, and the tangent line to $x^2 + 2y^2 = k$ has slope $\frac{dy}{dx}$, where $2x + 4y\frac{dy}{dx} = 0$, so $\boxed{B = \frac{dy}{dx} = -\frac{x}{2y}}$.
   But when the two curves intersect, we have in particular $y = cx^2$, so B becomes: $B = -\frac{x}{2cx^2} = -\frac{1}{2cx} = -\frac{1}{A}$, so when the curves intersect, the tangent lines to the two curves are perpendicular, so the two curves are orthogonal!

10. Don't worry about this one, Newton's method won't be on the exam!

11. $f'(x) = -\cos(x) + A$, but $0 = f'(0) = -1 + A$, so $A = 1$, and $f'(x) = -\cos(x) + 1$. Hence $f(x) = -\sin(x) + x + B$, but $1 = f(0) = 0 + 0 + B$, so $B = 1$, and $\boxed{f(x) = -\sin(x) + x + 1}$